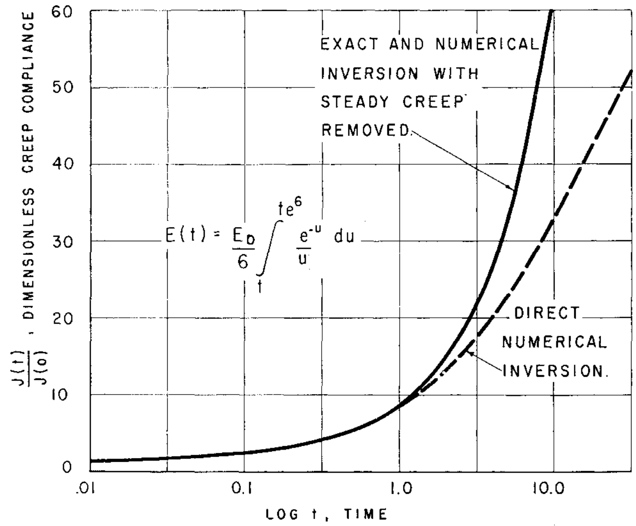

Fig. 1 Creep compliance with secondary creep.

The method herein presented may be extended and applied to any Volterra integral equation in which the form of the long-time response of the unknown function is known.

References

[1] Hopkins, I. L. and Hamming, R. W., "On creep and relaxation," J. Appl. Phys. **28**, 906–909 (1957).

[2] Lee, E. H. and Rogers, T. G., "Solution of viscoelastic stress analysis problems using measured creep of relaxation functions," J. Appl. Mech. **30**, 127 (1963).

[3] Dillon, O. W., "Transient stresses in non-homogeneous viscoelastic materials," J. Aerospace Sci. **29**, 284–288 (1962).

[4] Sackman, J. L., "A remark on transient stresses in non-homogeneous viscoelastic materials," J. Aerospace Sci. **29**, 1065 (1963).

[5] Taylor, R. L., "Problems in thermoviscoelasticity," Ph.D. Dissertation, Univ. of California, Berkeley, Calif. (June 1963).

Hypersonic Viscous Flow Near a Sharp Leading Edge

Richard W. Garvine*
Princeton University, Princeton, N. J.

Nomenclature

M = u_∞/a_∞ = freestream Mach number
Re_x = $\rho_\infty u_\infty x/\mu_\infty$ = Reynolds number
ϵ = $(\gamma - 1)/(\gamma + 1)$
λ = mean free path

Subscripts

∞ = conditions in freestream
s = conditions immediately behind shock
w = conditions at wall
0 = stagnation conditions

THE hypersonic viscous flow near the sharp leading edge of a flat plate has been the subject of many studies in recent years. Several investigators[1, 2] have reported measurements of surface pressure and heat transfer which begin to deviate markedly from the values predicted by viscous strong interaction theory as the leading edge is approached. Pressure and heat-transfer plateaus have been observed with levels well below those corresponding to strong interaction theory. The purpose of this note is to suggest that, when $(\epsilon M^2)^{1/2} \gg 1$, as in many cases of interest, these departures are not attributable to slip phenomena (except very near the leading edge) but rather to a viscous flow of more complicated nature than that of the strong interaction region. The mass flow is expected to divide nearly equally between a thin inviscid layer near the shock and a thick viscous layer below, across which the pressure is not constant and where the boundary layer approximation is not valid.

For a hypersonic flow, Hayes and Probstein (Chap. 10 of Ref. 3) use a mass conservation argument to point out that, upstream of the strong interaction region, the shock wave should become straight. Good evidence for this argument appears in the experimental work of Chuan and Waiter.[4] The shock inclination should be of the order $(\rho_\infty/\rho_s)^{1/2}$ or, for a perfect gas, $y_s/x \sim \epsilon^{1/2}$ (see Fig. 1). If a strong interaction region exists downstream, the shock wave in this region must have a large normal Mach number. The square of the normal Mach number in this straight shock or viscous layer region (as it will be termed in this note) is of the order $(My_s/x)^2 \sim \epsilon M^2$. A condition for the existence of this region is then $\epsilon M^2 \gg 1$. The pressure in the region should be reasonably uniform (since the shock is nearly straight) and of the order $p_\infty \epsilon M^2$. The upstream boundary of this region should be where noncontinuum phenomena begin to dominate the flow, whereas the downstream limitation of the viscous layer region should occur where strong interaction theory becomes valid. The pressure level for strong interaction flow is of the order $p_\infty \epsilon M^3/Re_x^{1/2}$, whereas that for the viscous layer flow is of the order $p_\infty \epsilon M^2$. Since the strong interaction pressure should always be less than the viscous layer pressure, the boundary between the two regions should occur for $M/Re_x^{1/2} \lesssim O(1)$. It is shown below that the upstream limit for the viscous layer region should occur where $M^2/Re_x \lesssim O(\epsilon M^2)^{1/2}$. A value for $M/Re_x^{1/2}$ of order unity ought then to characterize the viscous layer region.

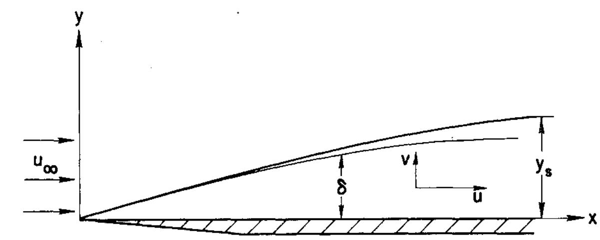

Fig. 1 Coordinate system for leading edge flow.

From schlieren photographs[1] the viscous layer region appears to have a viscous zone that nearly fills the entire shock layer. In this viscous zone, severe dissipation heats the fluid to nearly stagnation temperatures. Then the average temperature there would be of the order $T_\infty \epsilon M^2$. Since the pressure level is of the order $p_\infty \epsilon M^2$, the density will be of order ρ_∞ except very close to a cold wall.

Assuming for the moment that a thin shock adequately described by the Rankine-Hugoniot relations exists, there will be a thin zone of relatively cool inviscid flow between the shock and the viscous zone where the state properties are essentially those found immediately behind the shock. Thus, the density here will be of the order ρ_∞/ϵ, whereas the horizontal velocity will be of the order u_∞.

Over-all mass conservation requires the following relation:

$$\rho_\infty u_\infty y_s = \int_0^\delta \rho u dy + \int_\delta^{y_s} \rho u dy \qquad (1)$$

Received May 21, 1964. The research reported in this paper was supported by the Air Force Office of Scientific Research Grant AF 112-63. The author wishes to express his appreciation for the assistance given by S. H. Lam of the Department of Aerospace and Mechanical Sciences.

* Graduate Student and Assistant in Research, Department of Aerospace and Mechanical Sciences.

Here δ will be taken to characterize the extent of the viscous zone where the gas is strongly heated. In this zone, it can be expected that $u \sim u_\infty y/\delta$. Employing the preceding estimates in the integrals of Eq. (1), one obtains a relation between y_s and $y_s - \delta$ for over-all mass conservation:

$$(y_s - \delta)/y_s \sim \epsilon$$

The inviscid region would then appear quite thin in schlieren photographs, since $\epsilon \ll 1$ for most flows of interest. However, in terms of stream function or some other mass flow coordinate, the inviscid zone is not thin. Indeed, if one uses the previous estimates to compare the amounts of fluid mass at any instant in the two zones, one finds

$$\int_0^\delta \rho u dy \sim \int_\delta^{y_s} \rho u dy$$

so that the amounts of fluid in each zone are of the same order.

Estimates of the rarefaction effects in the viscous layer region are necessary if one is to apply continuum equations with any confidence there. The maximum value of the mean free path will occur where the density is least, i.e., in the hottest portion of the viscous zone, where $\rho \sim \rho_\infty$. Then $\lambda_{max} \sim \lambda_\infty$ so that, using the kinetic theory argument for viscosity,

$$\lambda_{max}/y_s \sim \lambda_\infty/(\epsilon^{1/2}x) \sim (\epsilon M^2)^{-1/2} M^2/Re_x$$

Then, since $M^2/Re_x \sim 1$, the requirement that $\lambda_{max}/y_s \ll 1$ is equivalent to the requirement $(\epsilon M^2)^{1/2} \gg 1$. This condition is slightly more stringent than that required of the square of the normal Mach number $(\epsilon M^2) \gg 1$.

The shock thickness should be of the order $\lambda_s \sim \epsilon\lambda_\infty$. Then, since $y_s - \delta \sim \epsilon^{3/2}x$, one finds

$$\lambda_s/(y_s - \delta) \sim (\epsilon M^2)^{-1/2} M^2/Re_x$$

The shock will then appear thin on the scale of the inviscid zone so long as $(\epsilon M^2)^{1/2} \gg 1$.

Velocity slip and temperature jump on the plate may be estimated in a similar fashion employing the usual slip boundary conditions. Using $(\partial u/\partial y)_w \sim u_\infty/\delta$ where $\delta \sim \epsilon^{1/2}x$ and $(\partial T/\partial y)_w \sim (T_0 - T_w)/\delta$, one finds

$$u_{slip}/u_\infty \sim (\lambda_w/u_\infty)(\partial u/\partial y)_w \sim (T_w/T_0)(\epsilon M^2)^{-1/2} M^2/Re_x$$

and

$$\Delta T_{jump}/T_0 \sim (\lambda_w/T_0)(\partial T/\partial y)_w \sim (T_w/T_0)[1 - (T_w/T_0)]\,(\epsilon M^2)^{-1/2} M^2/Re_x$$

Surface slip effects should then be small so long as $(\epsilon M^2)^{1/2} \gg 1$, especially for a cold wall. In general, rarefaction effects should be of higher order so long as $M^2/Re_x \sim 1$ and $(\epsilon M^2)^{1/2} \gg 1$. The upstream limit of the viscous layer region would appear to be where M^2/Re_x increases to a value less than, but of the order $(\epsilon M^2)^{1/2}$.

When the Navier-Stokes equations are applied to the viscous zone of the viscous layer region together with appropriately scaled variables, it will appear that the vertical momentum equation cannot be reduced to the simple boundary-layer form. With the scaling,

$$y = \epsilon^{1/2}\eta \qquad p = \epsilon M^2 p_\infty P$$

$$u = u_\infty U \qquad \rho = \rho_\infty R$$

$$v = \epsilon^{1/2} u_\infty V \qquad h = \epsilon M^2 h_\infty H$$

$$\mu = \epsilon M^2 \mu_\infty \mu^*$$

the dimensionless variables U, V, etc. should be of order unity in the viscous zone while the edge of the zone is near $\eta \sim x$. For simplicity, a perfect gas may be chosen with constant specific heats.

When these variables are substituted in the full Navier-Stokes equations, with terms of order ϵ compared to unity neglected, the following system of equations appears:

$$\frac{\partial}{\partial x}(RU) + \frac{\partial}{\partial \eta}(RV) = 0 \tag{2}$$

$$RU\frac{\partial U}{\partial x} + RV\frac{\partial U}{\partial \eta} = \frac{M^2}{Re_x} x \frac{\partial}{\partial \eta}\left(\mu^* \frac{\partial U}{\partial \eta}\right) \tag{3}$$

$$RU\frac{\partial V}{\partial x} + RV\frac{\partial V}{\partial \eta} = -\frac{1}{\gamma}\frac{\partial P}{\partial \eta} + \frac{M^2}{Re_x} x \left\{\frac{\partial}{\partial x}\left(\mu^*\frac{\partial U}{\partial \eta}\right) + \frac{4}{3}\frac{\partial}{\partial \eta}\left[\mu^*\left(\frac{\partial V}{\partial \eta} - \frac{1}{2}\frac{\partial U}{\partial x}\right)\right]\right\} \tag{4}$$

$$RU\frac{\partial H}{\partial x} + RV\frac{\partial H}{\partial \eta} = \frac{M^2}{Re_x} x \left[\frac{1}{Pr}\frac{\partial}{\partial \eta}\left(\mu^*\frac{\partial H}{\partial \eta}\right) + 2\mu^*\left(\frac{\partial U}{\partial \eta}\right)^2\right] \tag{5}$$

$$P = RH \tag{6}$$

Equations (2, 3, 5, and 6) are of the same form as the classical boundary-layer equations for a flat plate, but Eq. (4), the vertical momentum equation, is decidedly not of boundary-layer type, since $\partial P/\partial \eta$ is not small, and the equation is still of second order owing to the remaining viscous terms on the right-hand side. From this viewpoint, there appears no justification for the treatment of this flow with the boundary-layer equations as in Oguchi's analysis.[5] In fact, Hayes and Probstein, in Chap. 9 of Ref. 3, show, with an argument from mass conservation, that the assumption that the viscous zone extends to the shock (as Oguchi assumes) is incompatible with the boundary-layer concept.

It would appear, therefore, that a rational analysis of the viscous layer region must treat the unconventional nature of the vertical momentum equation. The variation in pressure across the layer should be retained, and, since the equation is of second order in η, the boundary condition for v at the viscous zone edge can no longer be ignored as in conventional boundary-layer analyses. The degree of approximation involved in neglecting rarefaction effects, i.e., dropping terms of order $(\epsilon M^2)^{-1/2}$, should be no worse than that involved in neglecting terms of order ϵ in the Navier-Stokes equations since, for many cases of practical interest, these terms are small and of the same order.

In conclusion, it is believed that conditions at the upstream edge of the viscous layer region have a crucial effect on the structure of the viscous layer region. It is felt that the leveling off of the heat-transfer values near the leading edge[2] is due in large part to the flow's memory of its passage through the noncontinuum region upstream.

References

[1] Nagamatsu, H. T. and Sheer, R. E., "Hypersonic shock wave-boundary layer interaction and leading edge slip," ARS J. **30**, 454–462 (1960).

[2] Vidal, R. J., Golian, T. C., and Bartz, J. A., "An experimental study of hypersonic low-density viscous effects on a sharp flat plate," AIAA Preprint 63-435 (August 1963).

[3] Hayes, W. D. and Probstein, R., *Hypersonic Flow Theory* (Academic Press Inc., New York, 1959), Chaps. 9 and 10.

[4] Chuan, R. L. and Waiter, S. A., "Experimental study of hypersonic rarefied flow near the leading edge of a thin flat plate," *Rarefied Gas Dynamics* (Academic Press Inc., New York, 1963), Sec. 5, pp. 328–342.

[5] Oguchi, H., "The sharp leading edge problem in hypersonic flow," *Rarefied Gas Dynamics* (Academic Press Inc., New York, 1961), Sec. 5, pp. 501–524.